

# Dilations of semigroups on von Neumann algebras and noncommutative $L^p$ -spaces

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## Abstract

We prove that any  $w^*$ -continuous semigroup of factorizable Markov maps acting on a von Neumann algebra  $M$  equipped with a state can be dilated by a group of Markov  $*$ -automorphisms in a manner analogous to the discrete case of one factorizable Markov operator. We also give a version of this result for strongly continuous semigroups of operators acting on noncommutative  $L^p$ -spaces, examples of semigroups to which the results of this paper can be applied and applications of these results to functional calculus of the generators of these semigroups.

## 1 Introduction

The study of dilations of operators is of central importance in operator theory and has a long tradition in functional analysis. Suppose  $1 < p < \infty$ . A classical result from seventies essentially due to Akcoglu [AkS] (see also [Pel]) says that a positive contraction  $T: L^p(\Omega) \rightarrow L^p(\Omega)$  on an  $L^p$ -space  $L^p(\Omega)$  admits a positive isometric dilation  $U$  on a bigger  $L^p$ -space than the initial  $L^p$ -space, i.e. there exists another measure space  $\Omega'$ , two positive contractions  $J: L^p(\Omega) \rightarrow L^p(\Omega')$  and  $P: L^p(\Omega') \rightarrow L^p(\Omega)$  and a positive invertible isometry  $U: L^p(\Omega') \rightarrow L^p(\Omega')$  such that  $T^k = PU^k J$  for any integer  $k \geq 0$ . Note that in this situation,  $J$  is an isometric embedding whereas  $JP$  is a contractive projection.

Later, Fendler [Fen1] proved a continuous version of this result for any strongly continuous semigroup  $(T_t)_{t \geq 0}$  of positive contractions on an  $L^p$ -space  $L^p(\Omega)$ . More precisely, this theorem says that there exists a measure space  $\Omega'$ , two positive contractions  $J: L^p(\Omega) \rightarrow L^p(\Omega')$  and  $P: L^p(\Omega') \rightarrow L^p(\Omega)$  and a strongly continuous group of positive invertible isometries  $(U_t)_{t \in \mathbb{R}}$  on  $L^p(\Omega')$  such that  $T_t = PU_t J$  for any  $t \geq 0$ , see also [Fen2].

In the noncommutative setting, measure spaces and  $L^p$ -spaces are replaced by von Neumann algebras and noncommutative  $L^p$ -spaces and positive maps by completely positive maps. In their remarkable paper [JLM], Junge and Le Merdy showed that there exists no “reasonable” analog of Akcoglu result for completely positive contractions acting on noncommutative  $L^p$ -spaces. It is a striking difference with the world of classical (=commutative)  $L^p$ -spaces of measure spaces.

Independently, Kümmerer, Maassen, Haagerup and Musat introduced and studied dilations of well-behaved completely positive unital operators on noncommutative probability spaces (=von Neumann algebras equipped with states), the so-called Markov operators [Kum1] [Kum2]

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[Kum3] [KuM] [HaM] [HaM2]. These dilations induce dilations on the associated noncommutative  $L^p$ -spaces. The following definition of these operators is considered in [AnD], [HaM] and [Ric].

**Definition 1.1** *Let  $(M, \phi)$  and  $(N, \psi)$  be von Neumann algebras equipped with normal faithful states  $\phi$  and  $\psi$ , respectively. A linear map  $T: M \rightarrow N$  is called a  $(\phi, \psi)$ -Markov map if*

- (1)  *$T$  is completely positive*
- (2)  *$T$  is unital*
- (3)  *$\psi \circ T = \phi$*
- (4)  *$T \circ \sigma_t^\phi = \sigma_t^\psi \circ T$ , for all  $t \in \mathbb{R}$ , where  $(\sigma_t^\phi)_{t \in \mathbb{R}}$  and  $(\sigma_t^\psi)_{t \in \mathbb{R}}$  denote the automorphism groups of the states  $\phi$  and  $\psi$ , respectively.*

In particular, when  $(M, \phi) = (N, \psi)$ , we say that  $T$  is a  $\phi$ -Markov map. Such an operator  $T$  induces a contraction  $T: L^p(M) \rightarrow L^p(M)$  on the associated noncommutative  $L^p$ -space  $L^p(M)$  for any  $1 \leq p < \infty$ , see for example [AnD, lemma 2.4].

The following definition is essentially due to Kümmerer (see [Kum2, Definition 2.1.1]). We refer to [Kum1] [Kum2] [Kum3] [KuM] for physical interpretations of this notion.

**Definition 1.2** *Let  $M$  be a von Neumann algebra with a normal faithful finite state  $\phi$  and let  $T: M \rightarrow M$  be a  $\phi$ -Markov map. We say that  $T$  is dilatable if there exists a von Neumann algebra  $N$  with a normal faithful state  $\psi$ , a  $*$ -automorphism  $U$  of  $N$  leaving  $\psi$  invariant and a  $(\phi, \psi)$ -Markov  $*$ -monomorphism  $J: M \rightarrow N$  satisfying*

$$T^k = \mathbb{E} U^k J, \quad k \geq 0.$$

where  $\mathbb{E} = J^*: N \rightarrow M$  is the canonical faithful normal conditional expectation preserving the states associated with  $J$ .

Note that Haagerup and Musat [HaM, Theorem 4.4] have succeeded in characterizing dilatable Markov maps. Indeed, they proved that a  $\phi$ -Markov map  $T$  is dilatable if and only if  $T$  is factorizable in the sense of [AnD], i.e. there exists a von Neumann algebra  $N$  equipped with a faithful normal state  $\psi$ , and  $*$ -monomorphisms  $J_0: M \rightarrow N$  and  $J_1: M \rightarrow N$  such that  $J_0$  is  $(\phi, \psi)$ -Markov and  $J_1$  is  $(\phi, \psi)$ -Markov, satisfying, moreover,  $T = J_0^* \circ J_1$ .

Now, we introduce the continuous version of this definition from [Arh2, Definition 1.3] inspired by Fendler result, see also [KuM, Definition page 4].

**Definition 1.3** *Let  $M$  be a von Neumann algebra equipped with a normal faithful state  $\phi$ . Let  $(T_t)_{t \geq 0}$  be a  $w^*$ -continuous semigroup of  $\phi$ -Markov maps on  $M$ . We say that the semigroup is dilatable if there exist a von Neumann algebra  $N$  equipped with a normal faithful state  $\psi$ , a  $w^*$ -continuous group  $(U_t)_{t \in \mathbb{R}}$  of  $*$ -automorphisms of  $N$ , a  $*$ -monomorphism  $J: M \rightarrow N$  such that each  $U_t$  is  $\psi$ -Markov and  $J$  is  $(\phi, \psi)$ -Markov satisfying*

$$(1.1) \quad T_t = \mathbb{E} U_t J, \quad t \geq 0,$$

where  $\mathbb{E} = J^*: N \rightarrow M$  is the canonical faithful normal conditional expectation preserving the states associated with  $J$ .

Note that such a dilation induces an isometric dilation similar to the one of Fendler theorem for the strongly continuous semigroup induced by the semigroup  $(T_t)_{t \geq 0}$  on the associated noncommutative  $L^p$ -space  $L^p(M)$  for any  $1 \leq p < \infty$ .

Our main result is the following theorem:

**Theorem 1.4** *let  $M$  be a von Neumann algebra equipped with a normal faithful state  $\phi$ . Let  $(T_t)_{t \geq 0}$  be a  $w^*$ -semigroup of factorizable  $\phi$ -Markov maps on  $M$ . Then the semigroup  $(T_t)_{t \geq 0}$  is dilatable.*

In particular, this result implies that all  $w^*$ -semigroups of selfadjoint Markov Fourier multipliers are dilatable, see Corollary 5.1. We also prove Theorem 4.4 which is a variant of this result for noncommutative  $L^p$ -spaces useful even for non- $\sigma$ -finite von Neumann algebras. Finally, we refer to the paper in preparation [JRS] for related results.

In the last section, we also give applications to  $H^\infty$  functional calculus which is a very useful and important tool in various areas: harmonic analysis of semigroups, multiplier theory, Kato's square root problem, maximal regularity in parabolic equations, control theory, etc. For detailed information we refer the reader to [Haa], [JMX] and [KW] and to the references therein.

The paper is organized as follows. The first section gives background Section 3 gives a proof of Theorem 1.4. In the following section 4, we describe and prove a noncommutative  $L^p$  analog of this result. In section 5, we give examples of dilatable semigroups. Finally, we conclude in section 6 with the applications of our results to functional calculus.

## 2 Preliminaries

**Noncommutative  $L^p$ -spaces** We use Haagerup noncommutative  $L^p$ -spaces. We refer to the survey [PiX] and to the papers [Ray2] and [Pis1] for more information.

**Markov operators** Note that a linear map  $T: M \rightarrow N$  satisfying conditions (1) – (3) of Definition 1.1 is automatically normal. If, moreover, condition (4) is satisfied, then it was proved in [AcC] (see also [AnD, Lemma 2.5]) that there exists a unique completely positive unital map  $T^*: N \rightarrow M$  such that

$$(2.1) \quad \phi(T^*(y)x) = \psi(yT(x)), \quad x \in M, y \in N.$$

It is easy to show that  $T^*$  is a  $(\psi, \phi)$ -Markov map. Moreover, we say that a  $\phi$ -Markov map  $T: M \rightarrow M$  is selfadjoint if  $T = T^*$ .

Finally, it is not difficult to prove that a  $(\phi, \psi)$ -Markov  $*$ -homomorphism is always injective. Indeed, let  $T: M \rightarrow N$  be a  $*$ -homomorphism preserving the states and consider  $x \in M^+$ . Suppose  $T(x) = 0$ . We have  $\phi(x) = \psi(T(x)) = 0$ . Hence  $x = 0$  by the positivity of  $T$  and the faithfulness of  $\phi$ . Now if  $y \in M$  satisfies  $T(y) = 0$ . We have  $T(y)^*T(y) = 0$ . Since  $T$  is a  $*$ -homomorphism, we infer that  $T(y^*y) = 0$ . We deduce that  $y^*y = 0$  and therefore that  $y = 0$ .

**Ultraproducts of Banach spaces** Let  $(X_n)_{n \geq 1}$  be a sequence of Banach spaces, and let  $\ell^\infty(\mathbb{N}, X_n)$  be the Banach space of all sequences  $(x_n)_{n \geq 1} \in \prod_{n=1}^\infty X_n$  with  $\sup_{n \geq 1} \|x_n\|_{X_n} < \infty$  with the norm  $\|(x_n)_{n \geq 1}\| = \sup_{n \geq 1} \|x_n\|_{X_n}$ . Let  $\mathcal{U}$  be a free ultrafilter on  $\mathbb{N}$ . The Banach space ultraproduct  $(X_n)^\mathcal{U}$  is defined as the quotient  $\ell^\infty(\mathbb{N}, X_n)/\mathcal{J}_\mathcal{U}$ , where  $\mathcal{J}_\mathcal{U}$  is the closed subspace of all  $(x_n)_{n \geq 1} \in \ell^\infty(\mathbb{N}, X_n)$  which satisfies  $\lim_{n \rightarrow \mathcal{U}} \|x_n\|_{X_n} = 0$ . An element of  $(X_n)^\mathcal{U}$  represented by  $(x_n)_{n \geq 1} \in \ell^\infty(\mathbb{N}, E)$  is written as  $(x_n)^\mathcal{U}$ . For any  $(x_n)^\mathcal{U} \in (X_n)^\mathcal{U}$ , one has  $\|(x_n)^\mathcal{U}\| = \lim_{n \rightarrow \mathcal{U}} \|x_n\|_{X_n}$ . If  $(T_n: X_n \rightarrow Y_n)_{n \geq 1}$  is a bounded sequence of bounded linear operators, we can define the ultraproduct map  $T: (X_n)^\mathcal{U} \rightarrow (Y_n)^\mathcal{U}$ ,  $(x_n)^\mathcal{U} \mapsto (T_n(x_n))^\mathcal{U}$ . We refer to [DJT, section 8] for more information.

If  $1 < p < \infty$ , an ultraproduct of noncommutative  $L^p$ -spaces is a noncommutative  $L^p$ -space, see [Ray1]. However, the Banach space ultraproduct of von Neumann algebras is not a von Neumann algebra in general.

**Ultraproducts of von Neumann algebras** If  $\phi$  is a normal faithful state on a von Neumann algebra  $M$ , we define  $\|\cdot\|_\phi^\sharp$  by

$$\|x\|_\phi^\sharp = \phi(x^*x + xx^*)^{\frac{1}{2}}, \quad x \in M.$$

Let us now define the (Ocneanu) ultraproduct  $(M_n, \phi_n)^\mathcal{U}$  of a sequence  $(M_n, \phi_n)_{n \geq 1}$  of  $\sigma$ -finite von Neumann algebras equipped with normal faithful states  $\phi_n$  with respect to a free ultrafilter  $\mathcal{U}$  over  $\mathbb{N}$ . Define  $\ell^\infty(\mathbb{N}, M_n)$  the C\*-algebra of sequences  $(x_n)_{n \geq 1} \in \prod_{n=1}^\infty M_n$  such that  $\sup_{n \geq 1} \|x_n\|_{M_n} < +\infty$  endowed with the norm  $\|(x_n)\|_{\ell^\infty(\mathbb{N}, M_n)} = \sup_{n \geq 1} \|x_n\|_{M_n}$ . Let  $\mathcal{U}$  be free ultrafilter on  $\mathcal{N}$ . We let

$$\mathcal{I}_\mathcal{U}(M_n, \phi_n) := \left\{ (x_n)_{n \geq 1} \in \ell^\infty(\mathbb{N}, M_n) : \|x_n\|_\phi^\sharp \xrightarrow[n \rightarrow \mathcal{U}]{} 0 \right\},$$

and also, with the abbreviated notation  $\mathcal{I}_\mathcal{U}$  for  $\mathcal{I}_\mathcal{U}(M_n, \phi_n)$ , let

$$\mathcal{M}^\mathcal{U}(M_n, \phi_n) := \left\{ (x_n)_{n \geq 1} \in \ell^\infty(\mathbb{N}, M_n) : (x_n)_n \mathcal{I}_\mathcal{U} \subset \mathcal{I}_\mathcal{U}, \text{ and } \mathcal{I}_\mathcal{U}(x_n)_n \subset \mathcal{I}_\mathcal{U} \right\}.$$

It is then apparent that  $\mathcal{M}^\mathcal{U}(M_n, \phi_n)$  is a C\*-algebra (with pointwise operations and supremum norm) in which  $\mathcal{I}_\mathcal{U}(M_n, \phi_n)$  is a closed ideal. We then define

$$(M_n, \phi_n)^\mathcal{U} := \mathcal{M}^\mathcal{U}(M_n, \phi_n) / \mathcal{I}_\mathcal{U}(M_n, \phi_n)$$

(the quotient C\*-algebra). Moreover,  $(M_n, \phi_n)^\mathcal{U}$  is a W\*-algebra. We denote the image of  $(x_n)_{n \geq 1} \in \mathcal{M}^\mathcal{U}(M_n, \phi_n)$  in  $(M_n, \phi_n)^\mathcal{U}$  as  $(x_n)^\mathcal{U}$ . Finally, the following defines a normal faithful state  $(\phi_n)^\mathcal{U}$  on  $(M_n, \phi_n)^\mathcal{U}$ :

$$(\phi_n)^\mathcal{U}((x_n)^\mathcal{U}) := \lim_{n \rightarrow \mathcal{U}} \phi_n(x_n), \quad (x_n)^\mathcal{U} \in (M_n, \phi_n)^\mathcal{U}.$$

See [AHW], [AnH] and [Pis2, Section 9.10] for more information.

Similarly to [Ued, page 352], if  $(J_n: M_n \rightarrow N_n)_{n \geq 1}$  is a sequence of  $(\phi_n, \psi_n)$ -Markov \*-monomorphism then it is not difficult to prove that we can define the ultraproduct map  $(J_n)^\mathcal{U}: (M_n)^\mathcal{U} \rightarrow (N_n)^\mathcal{U}$ ,  $(x_n)^\mathcal{U} \mapsto (J_n(x_n))^\mathcal{U}$  which is a  $((\phi_n)^\mathcal{U}, (\psi_n)^\mathcal{U})$ -Markov \*-monomorphism. Indeed the  $J_n$ 's induce a \*-homomorphism  $J = \oplus J_n: \ell^\infty(\mathbb{N}, M_n) \rightarrow \ell^\infty(\mathbb{N}, N_n)$ . Moreover it is easy to check that if  $(x_n)_{n \geq 1} \in \mathcal{M}^\mathcal{U}(M_n, \phi_n)$  then  $(J_n(x_n))_{n \geq 1} \in \mathcal{M}^\mathcal{U}(N_n, \psi_n)$  and if  $(x_n)_{n \geq 1} \in \mathcal{I}_\mathcal{U}(M_n, \phi_n)$  then  $(J_n(x_n))_{n \geq 1} \in \mathcal{I}_\mathcal{U}(N_n, \psi_n)$ . The restriction of  $\oplus J_n$  gives a map  $\mathcal{M}^\mathcal{U}(M_n, \phi_n) \rightarrow \mathcal{M}^\mathcal{U}(N_n, \psi_n)$ . The quotient map is the ultraproduct map  $(J_n)^\mathcal{U}$ . Moreover, if  $\mathbb{E}_n: N_n \rightarrow M_n$  is the canonical faithful normal conditional expectation preserving the states associated with  $J_n$  then the equation

$$(\mathbb{E}_n)^\mathcal{U}((x_n)^\mathcal{U}) = (\mathbb{E}_n(x_n))^\mathcal{U}$$

gives rise to a well-defined normal faithful conditional expectation  $(\mathbb{E}_n)^\mathcal{U}: (N_n)^\mathcal{U} \rightarrow (M_n)^\mathcal{U}$  such that  $(\phi_n)^\mathcal{U} \circ (\mathbb{E}_n)^\mathcal{U} = (\psi_n)^\mathcal{U}$ .

**Convexity** A normed linear space  $X$  is locally uniformly convex if for any  $\varepsilon > 0$  and any  $x \in X$  with  $\|x\| = 1$  there exists  $\delta(\varepsilon, x) > 0$  such that  $\|y\| = 1$  and  $\frac{\|x+y\|}{2} \geq 1 - \delta(\varepsilon, x)$  imply  $\|y - x\| \leq \varepsilon$ . It is clear from the definition that uniform convexity implies local uniform convexity.

**Semi-groups of operators** Let  $X$  be a Banach space. Recall that a semigroup  $(T_t)_{t \geq 0}$  of operators on  $X$  is strongly continuous if the map  $t \mapsto T_t x$  is continuous from  $\mathbb{R}^+$  into  $X$  for any  $x \in X$ .

Let  $X$  be a dual Banach space with predual  $X_*$ . We say that a semigroup  $(T_t)_{t \geq 0}$  of operators on  $X$  is  $w^*$ -continuous if the map  $t \mapsto \langle y, T_t x \rangle_{X_*, X}$  is continuous on  $\mathbb{R}^+$  for any  $x \in X$  and any  $y \in X_*$ . In passing, note that the weak\* topology on the Banach space  $B(X)$  is the topology of pointwise convergence on  $X$  endowed with the  $\sigma(X, X_*)$ -topology<sup>1</sup>.

**Representations of groups and kernels** Let  $X$  be a Banach space. Let  $\pi: G \rightarrow B(X)$  be a representation of a group  $G$  on  $X$ . Then we say that  $\pi$  is bounded when  $\sup \{\|\pi_t\| : t \in G\} < \infty$ .

We need some notions and results of the papers [DLG1] and [DLG2]. Recall that a non-empty subset  $D$  of an algebraic semigroup  $\mathcal{S}$  is called a two-sided ideal if  $\mathcal{S}D \subset D$  and if  $D\mathcal{S} \subset D$ . If  $\mathcal{S}$  is a semigroup, the intersection of all the two-sided ideals of  $\mathcal{S}$  is called the kernel of  $\mathcal{S}$ . If  $\mathcal{S}$  be a compact (Hausdorff) semitopological semigroup, that is a semigroup with separately continuous semigroup operations, then it is known [DLG1, Theorem 2.3] that its kernel is non-empty.

Let  $\pi: G \rightarrow B(X)$  be a (non-continuous) bounded representation of a topological group  $G$  on a reflexive Banach space  $X$ . Then we denote by

$$X_c = \{x \in X : t \mapsto \pi_t x \text{ is continuous from } G \text{ to } X\}$$

the subspace of continuously translating elements of  $X$  for the representation  $\pi$ . Let  $\mathcal{V}(e)$  be the set of all neighbourhoods  $V$  of the identity  $e$  of  $G$ . We then set  $\mathcal{S}^c(\pi)$  be the closure in the weak operator topology of the convex hull of  $\bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_t : t \in V\}}^{wo}$ , endowed with the weak operator topology and called the convex semigroup of  $\pi$  over the identity  $e$ . Then it is known [DLG2, Lemma 2.3] that  $\mathcal{S}^c(\pi)$  is a compact semitopological semigroup. A consequence of [DLG1, Theorem 7.2] is that the kernel  $\mathcal{K}(\pi)$  of  $\mathcal{S}^c(\pi)$  consists entirely of projections. The result [DLG2, Lemma 2.4] (and the remarks before) gives the following result.

**Theorem 2.1** *Let  $X$  be a reflexive Banach space and  $\pi: G \rightarrow B(X)$  be a (non-continuous) bounded representation of a commutative topological group  $G$ . Then the kernel  $\mathcal{K}(\pi)$  of the convex semigroup  $\mathcal{S}^c(\pi)$  of  $\pi$  contains a unique idempotent  $Q$  and  $Q$  is a bounded projection of  $X$  on  $X_c$  with  $Q\pi_t = \pi_t Q$  for any  $t \in G$ .*

**Accumulation points** Let  $(x_i)_{i \in I}$  be a net in a topological space  $X$ . An accumulation point of the net  $(x_i)_{i \in I}$  is an element of the intersection  $\bigcap_{F \in \mathcal{F}} \overline{F}$  where

$$\mathcal{F} = \{F \subset X : \text{there exists } i_0 \in I \text{ such that } \{x_i : i \geq i_0\} \subset F\}$$

or equivalently a limit of some subnet of  $(x_i)_{i \in I}$ .

### 3 Dilations of semigroups on von Neumann algebras

Suppose that  $X$  is a dual Banach space  $X$  with predual  $X_*$ . It is well-known [BJM, Exercise 1.12 page 251] that the space  $B_{w^*}(X)$  of weak\* continuous operator of  $B(X)$  is a semitopological semigroup with respect to the weak\* topology and that the mapping

$$(3.1) \quad \begin{array}{ccc} B(X_*) & \longrightarrow & B_{w^*}(X) \\ T & \longmapsto & T^* \end{array}$$

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1. A net  $(T_i)$  in  $B(X)$  converges to a point  $T \in B(X)$  in the weak\* topology if and only if for any  $x \in X$  and any  $y \in X_*$  we have  $\langle y, T_i x \rangle_{X_*, X} \xrightarrow{i} \langle y, T x \rangle_{X_*, X}$ .

is a weak operator-weak\* homeomorphism. Moreover, it is easy to prove that  $B_{w^*}(X)$  is a closed subspace of  $B(X)$ .

Let  $G$  be a topological group and let  $\pi: G \rightarrow B_{w^*}(X)$  be a (non-continuous) bounded representation on a dual Banach space  $X$  by weak\* continuous operators. Then, we define

$$X_{w^*} = \left\{ x \in X : t \mapsto \langle y, \pi_t x \rangle_{X^*, X} \text{ is continuous from } G \text{ to } \mathbb{C} \text{ for any } y \in X_* \right\}$$

called the subspace of weak\* continuously translating elements of  $X$ . Let  $\mathcal{V}(e)$  be the set of all neighbourhoods  $V$  of the identity  $e$  of  $G$ . We then set  $\mathcal{S}^{w^*}(\pi)$  be the closure in the weak\* topology of  $B(X)$  of the convex hull of  $\bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_t : t \in V\}}^{w^*}$ , endowed with the weak\* operator topology where the closure (or equivalently in  $B_{w^*}(X)$ ). Note that each set  $\overline{\{\pi_t : t \in V\}}^{w^*}$  is a subset of  $B_{w^*}(X)$ .

The following three propositions are weak\* analogs of the results of [DLG2, lemma 2.3] and [DLG2, lemma 2.4].

**Lemma 3.1** *Let  $\pi: G \rightarrow B_{w^*}(X)$  be a bounded (non-continuous) representation of a topological group  $G$  on a dual Banach space  $X$  such that each  $\pi_t$  is  $w^*$ -continuous for any  $t \in G$ . The sets  $\bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_t : t \in V\}}^{w^*}$  and  $\mathcal{S}^{w^*}(\pi)$  are compact semitopological semigroups.*

*Proof :* The subset  $\{\pi_t : t \in V\}$  of the dual Banach space  $B(M)$  is norm-bounded, hence the set  $\overline{\{\pi_t : t \in V\}}^{w^*}$  is compact for the weak\* topology by Alaoglu's theorem. We deduce that the intersection  $\bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_t : t \in V\}}^{w^*}$  is compact. Using [DS, Exercise 3, page 511] and the homeomorphism (3.1), we conclude that its closed convex hull  $\mathcal{S}^{w^*}(\pi)$  is also compact.

Let  $T$  and  $R$  be elements of  $\bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_t : t \in V\}}^{w^*}$ . Let  $U$  be a neighbourhood of  $R$  for the weak\* topology. For any neighbourhood  $V$  of  $e$ , we have  $W \cap \{\pi_t : t \in V\} \neq \emptyset$ . Hence there exists  $t_{V,U} \in V$  such that  $\pi_{t_{V,U}} \in U$ . The net  $(t_{V,U})^2$  converge to  $e$  in  $G$  and the net  $(\pi_{t_{V,U}})$  converge to  $R$  in the weak\* topology.

Let  $V$  be an element of  $\mathcal{V}(e)$ . Choose  $W$  in  $\mathcal{V}(e)$  such that  $W^2 \subset V$ . Note that  $T \in \overline{\{\pi_t : t \in W\}}^{w^*}$ . We have

$$\{\pi_t : t \in W\} \cdot \{\pi_t : t \in W\} \subset \{\pi_t : t \in V\} \subset \overline{\{\pi_t : t \in V\}}^{w^*}.$$

If  $V \subset W$ , i.e. if  $(W, U) \preceq (V, U)$ , we have  $t_{V,U} \in W$  and

$$T \cdot \pi_{t_{V,U}} \in \overline{\{\pi_t : t \in W\}}^{w^*} \cdot \{\pi_t : t \in W\} \subset \overline{\{\pi_t : t \in V\}}^{w^*}.$$

Passing to the limit, we deduce that  $TR \in \overline{\{\pi_t : t \in V\}}^{w^*}$  for any  $V \in \mathcal{V}(e)$ . Hence  $TR$  belongs to the set  $\bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_t : t \in V\}}^{w^*}$ , i.e. this latter set is a semigroup. Finally its closed convex hull  $\mathcal{S}^{w^*}(\pi)$  is clearly also a semigroup. ■

**Proposition 3.2** *Let  $\pi: G \rightarrow B_{w^*}(X)$  be a bounded (non-continuous) representation of a commutative topological group  $G$  on a dual Banach space  $X$  such that  $\pi_t$  is  $w^*$ -continuous for any  $t \in G$ . The kernel of the compact semitopological semigroup  $\mathcal{S}^{w^*}(\pi)$  contains a unique projection  $Q$  such that  $Q\pi_t = \pi_t Q$  for any  $t \in G$ .*

2. Declare that  $(V_1, U_1) \preceq (V_2, U_2)$  if  $V_2 \subset V_1$  and  $U_2 \subset U_1$ .

*Proof* : For any  $t \in G$ , we have

$$\begin{aligned} \pi_t \left( \bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_s : s \in V\}}^{w*} \right) \pi_{t^{-1}} &= \bigcap_{V \in \mathcal{V}(e)} \pi_t \overline{\{\pi_t : t \in V\}}^{w*} \pi_t^{-1} \\ &= \bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_s : s \in V\} \pi_t^{-1}}^{w*} = \bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_s : s \in V\}}^{w*}. \end{aligned}$$

So  $\sigma \mapsto \pi_t \sigma \pi_t^{-1}$  is an automorphism of the semigroup  $\mathcal{S}^{w*}(\pi)$ . But any automorphism of  $\mathcal{S}^{w*}(\pi)$  preserves the least ideal (the kernel). In particular, by uniqueness of the projection  $Q$ , we deduce that  $\pi_t Q \pi_t^{-1} = Q$  for any  $t \in G$ .  $\blacksquare$

**Proposition 3.3** *Let  $\pi : G \rightarrow B(X)$  be a bounded (non-continuous) representation of a topological group  $G$  on a dual Banach space  $X$ . The set  $X_{w*}$  consists of precisely those  $x$  in  $X$  which are fixed under all  $T$  in  $\bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_t : t \in V\}}^{w*}$  :*

$$X_{w*} = \left\{ x \in X : T(x) = x \text{ for any } T \in \bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_t : t \in V\}}^{w*} \right\}.$$

*Proof* : Consider  $x \in X_{w*}$ . If  $y \in X_*$  then for any  $\varepsilon > 0$ , using the continuity of  $t \mapsto \langle y, \pi_t x \rangle_{X_*, X}$  at  $e$ , we see that there exists a neighbourhood  $V_{\varepsilon, x, y} \in \mathcal{V}(e)$  such that for any  $t \in V_{\varepsilon, x, y}$

$$\left| \langle y, \pi_t x \rangle_{X_*, X} - \langle y, x \rangle_{X_*, X} \right| < \varepsilon.$$

Let  $\sigma$  be an element of the closure  $\overline{\{\pi_t : t \in V_{\varepsilon, x, y}\}}^{w*}$ . There exists a net  $(\pi_{t_i})_{i \in I}$  with  $t_i \in V_{\varepsilon, x, y}$  converging to  $\sigma$  in the weak\* topology. For any  $i \in I$ , we have

$$\left| \langle y, \pi_{t_i} x \rangle_{X_*, X} - \langle y, x \rangle_{X_*, X} \right| < \varepsilon.$$

Passing to the limit, we obtain

$$\left| \langle y, \sigma(x) - x \rangle_{X_*, X} \right| = \left| \langle y, \sigma(x) \rangle_{X_*, X} - \langle y, x \rangle_{X_*, X} \right| < \varepsilon.$$

Now, if  $\sigma_0 \in \bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_t : t \in V\}}^{w*}$  then for any  $\varepsilon > 0$  and any  $y \in X_*$  the element  $\sigma_0$  belongs to  $\overline{\{\pi_t : t \in V_{\varepsilon, x, y}\}}^{w*}$ . For any  $y \in X_*$ , we deduce that

$$\left| \langle y, \sigma_0(x) - x \rangle_{X_*, X} \right| = 0$$

We conclude that  $\sigma_0(x) = x$ .

For the reverse inclusion, let  $x \in X$  fixed by all elements of  $\bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_t : t \in V\}}^{w*}$ , i.e. suppose that for any  $\sigma \in \bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_t : t \in V\}}^{w*}$  we have  $\sigma(x) = x$ . Consider a net  $(t_i)_{i \in I}$  in  $G$  converging to the identity  $e$ . Since the representation  $\pi$  is bounded, the set  $\overline{\{\pi_t : t \in G\}}^{w*}$  is weak\* compact. Using the continuous map  $B(X) \mapsto X, T \mapsto Tx$ , where the spaces are equipped with the weak\* topology, we see that the subset  $\overline{\{\pi_t : t \in G\}}^{w*} \cdot x$  of  $X$  is compact for the weak\* topology.

Note that an accumulation point of the net  $(\pi_{t_i})_{i \in I}$  is an element of  $\bigcap_{F \in \mathcal{F}} \overline{F}$  where

$$\mathcal{F} = \{F \subset B(X) : \text{there exists } i_0 \in I \text{ such that } \{\pi_{t_i} : i \geq i_0\} \subset F\}.$$

For any neighbourhood  $V \in \mathcal{V}(e)$  there exists  $i_V$  such that  $i \geq i_V$  imply  $t_i \in V$  and thus  $\pi_{t_i} \in \pi(V)$ . Thus the set  $\{\pi(t_i) : i \geq i_V\}$  is included in  $\{\pi_t : t \in V\}$ . Then the set  $\{\pi_t : t \in V\}$  belongs to  $\mathcal{F}$ . We deduce that

$$\bigcap_{F \in \mathcal{F}} \overline{F}^{w^*} \subset \bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_t : t \in V\}}^{w^*}.$$

We conclude that the net  $(\pi_{t_i})_{i \in I}$  can have accumulation points only in the intersection  $\bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_t : t \in V\}}^{w^*}$ . Now, it is not difficult to see that the net  $(\pi_{t_i}x)_{i \in I}$  of  $X$  can only have accumulation points in  $\bigcap_{V \in \mathcal{V}(e)} \overline{\{\pi_t : t \in V\}}^{w^*} \cdot x = \{x\}$ . Lying in the weak\* compact subset  $\overline{\{\pi_t : t \in G\}}^{w^*} \cdot x$  of  $X$ , we infer that it converges weak\* to  $x$ . Consequently, the map  $t \mapsto \pi_t x$  is weak\* continuous at  $t = e$ , hence everywhere, completing the proof.  $\blacksquare$

The following result is a particular case of the combination of [BGKS, Theorem 1.2], [BGKS, Proposition 5.5], [BGKS, Remark 5.6] and [BGKS, Corollary 4.3] (and its proof), see also [KuN, Theorem 2.4]. Here, we use the fact that a unital completely positive map  $T: M \rightarrow M$  on a von Neumann algebra  $M$  is a Schwarz map [Pau, Proposition 3.3], i.e:

$$T(x)^*T(x) \leq T(x^*x), \quad x \in M.$$

**Theorem 3.4** *Let  $M$  be a von Neumann algebra equipped with a normal faithful state  $\phi$ . Let  $\mathcal{S}$  be a semigroup of normal unital completely positive maps  $T: M \rightarrow M$  leaving  $\phi$  invariant. The closure  $\overline{\text{co } \mathcal{S}}^{w^*}$  of the convex hull  $\text{co}(\mathcal{S})$  of  $\mathcal{S}$  in the weak\* topology of  $B(M)$  is a compact semitopological semigroup and its kernel is a singleton  $\{\mathbb{E}\}$  where  $\mathbb{E}$  is a faithful normal conditional expectation  $\mathbb{E}: M \rightarrow M$  leaving  $\phi$  invariant satisfying*

$$\text{Ran } \mathbb{E} = \{x \in M : T(x) = x \text{ for any } T \in \mathcal{S}\}.$$

The following lemma is a generalization of [Fen1, Lemma 3]. Thanks the uniform convexity of noncommutative  $L^p$ -spaces [PiX, Corollary 5.2], this lemma can be applied to noncommutative  $L^p$ -spaces.

**Lemma 3.5** *Let  $X$  be a Banach space and let  $Y$  be a locally uniformly convex Banach space. Let  $(T_t)_{t \geq 0}$  be a strongly continuous semigroup of contractions on  $X$ . Let  $(U_t)_{t \in \mathbb{Q}}$  be a (non continuous) group of isometries on  $X$  and  $J: X \rightarrow Y$  and  $P: Y \rightarrow X$  two contractions such that  $T_t = PU_tJ$  for any  $t \in \mathbb{Q}^+$ . If  $x \in X$  then the map*

$$\begin{array}{ccc} \mathbb{Q} & \longrightarrow & Y \\ t & \longmapsto & U_t Jx \end{array}$$

*is continuous from  $\mathbb{Q}$  to  $Y$  with its norm topology.*

*Proof :* Let  $x \in X$  with  $\|x\| = 1$ . Note that  $\|Jx\|_Y = \|x\|_X = 1$ . By the locally uniform convexity of  $Y$ , if  $\varepsilon > 0$  there exists  $\delta(\varepsilon, x) > 0$  such that if  $y \in Y$  satisfies  $\|y\|_Y = 1$  and  $\frac{\|y + Jx\|}{2} \geq 1 - \delta(\varepsilon, x)$  we have  $\|y - Jx\|_Y \leq \varepsilon$ . Since  $(U_t)_{t \in \mathbb{Q}}$  is a group of isometries, it suffices



to show, for  $x \in X$ , the continuity from the right of the map  $s \mapsto U_t Jx$  at  $t = 0$ . Given  $\varepsilon > 0$ , by the strong continuity of  $(T_t)_{t \in \mathbb{Q}}$  there exists  $\delta > 0$  such that  $0 \leq t \leq \delta$  imply  $\|T_t x - x\| \leq \delta(\varepsilon, x)$ . Hence, for any  $t \in \mathbb{Q} \cap [0, \delta]$  we have

$$\begin{aligned} \|U_t Jx + Jx\|_Y &\geq \|PU_t Jx + PJx\|_X = \|T_t x + x\|_X \\ &= \|2x - (x - T_t x)\|_X \geq \|2x\|_X - \|T_t x - x\|_X \geq 2 - 2\delta(\varepsilon, x). \end{aligned}$$

Hence  $\frac{\|U_t Jx + Jx\|_Y}{2} \geq 1 - \delta(\varepsilon, x)$ . Since  $\|U_t Jx\|_Y = \|Jx\|_Y = \|x\|_X = 1$ , we infer  $\|U_t Jx - Jx\|_Y \leq \varepsilon$ .  $\blacksquare$

The following lemma is a variant of the above Lemma.

**Lemma 3.6** *Let  $M$ , and  $N$  be von Neumann algebras equipped with normal faithful states  $\phi$  and  $\psi$ . Let  $(T_t)_{t \geq 0}$  be a  $w^*$ -continuous semigroup of  $\phi$ -Markov maps on  $M$ . Let  $(U_t)_{t \in \mathbb{Q}}$  be a group of  $*$ -automorphisms of  $N$  leaving  $\psi$  invariant and  $J: M \rightarrow N$  a  $(\phi, \psi)$ -Markov  $*$ -monomorphism such that  $T_t = \mathbb{E}U_t J$  for any  $t \in \mathbb{Q}^+$  where  $\mathbb{E}: N \rightarrow M$  is the canonical faithful normal conditional expectation preserving the states associated with  $J$ . For any  $x \in M$  and any  $y \in L^1(N)$ , the map*

$$\begin{array}{ccc} \mathbb{Q} & \longrightarrow & \mathbb{C} \\ t & \longmapsto & \langle y, U_t J(x) \rangle_{L^1(N), N} \end{array}$$

*is continuous.*

*Proof :* We fix  $1 < p < \infty$ . The semigroup  $(T_t)_{t \geq 0}$  induces a strongly continuous semigroup of contractions on  $L^p(M)$  and the semigroup  $(U_t)_{t \in \mathbb{Q}}$  induces a group of isometries on  $L^p(N)$ . Moreover  $J$  induces an isometric embedding of  $L^p(M)$  into  $L^p(N)$  and  $\mathbb{E}$  a contractive map from  $L^p(N)$  onto  $L^p(M)$ . For any  $x \in L^p(M)$ , by Lemma 3.5, the map  $t \rightarrow U_t Jx$  is continuous from  $\mathbb{Q}$  to  $L^p(N)$  with its norm topology. Let  $t_0 \in \mathbb{Q}$  and let  $D \in L^1(N)$  the density operator of  $\psi$ . Note that  $D^{\frac{1}{2p}} \in L^{2p}(N)$ . For any  $z \in L^{p^*}(N)$  and any  $x \in L^p(M)$ , we have

$$\begin{aligned} \langle D^{\frac{1}{2p}} z D^{\frac{1}{2p}}, U_t Jx \rangle_{L^1(N), N} &= \langle z, D^{\frac{1}{2p}} U_t Jx D^{\frac{1}{2p}} \rangle_{L^{p^*}(N), L^p(N)} \\ &\xrightarrow{t \rightarrow t_0} \langle z, D^{\frac{1}{2p}} U_{t_0} Jx D^{\frac{1}{2p}} \rangle_{L^{p^*}(N), L^p(N)} = \langle D^{\frac{1}{2p}} z D^{\frac{1}{2p}}, U_{t_0} Jx \rangle_{L^1(N), N}. \end{aligned}$$

Recall that  $D^{\frac{1}{2p}} L^{p^*}(N) D^{\frac{1}{2p}}$  is norm dense in  $L^1(N)$ . Now, with a  $\frac{\varepsilon}{3}$ -argument, it is not difficult to complete this proof.  $\blacksquare$

Now we can prove our first main result. We use a similar strategy to the one of Fendler [Fen1]. However, the method of [Fen1] does not apply identically to our context. We adapt instead some trick of the proof of [AFM, Corollary 6.2] using some results from the papers [DLG1] and [DLG2].

**Theorem 3.7** *let  $M$  be a von Neumann algebra equipped with a normal faithful state  $\phi$ . Let  $(T_t)_{t \geq 0}$  be a  $w^*$ -semigroup of factorizable  $\phi$ -Markov map on  $M$ . Then the semigroup  $(T_t)_{t \geq 0}$  is dilatable.*

*Proof :* For a finite set  $B \subset \mathbb{Q}$  let  $U_B = \{n \in \mathbb{N} : nt \in \mathbb{Z} \text{ for any } t \in B\}$ . Then the set of all sets  $\{U_B : B \subset \mathbb{Q}, B \text{ finite}\}$  is closed under finite intersections and thus constitutes the basis of some filter  $\mathcal{F}$  which is contained in some ultrafilter  $\mathcal{U}$ .

Using [HaM, Theorem 4.4], for any integer  $n \geq 0$ , we note that the operator  $T_{\frac{1}{n}}: M \rightarrow M$  is dilatable. This means that there exist a von Neumann algebra  $N_{\frac{1}{n}}$  equipped a normal

faithful state  $\psi_{\frac{1}{n}}$ , a  $*$ -automorphism  $S_{\frac{1}{n}}$  of  $N_n$  leaving  $\psi_{\frac{1}{n}}$  invariant and a  $(\phi, \psi_{\frac{1}{n}})$ -Markov  $*$ -monomorphism  $J: M \rightarrow N_{\frac{1}{n}}$  such that

$$(T_{\frac{1}{n}})^k = \mathbb{E}_{\frac{1}{n}}(S_{\frac{1}{n}})^k J_{\frac{1}{n}}, \quad k \geq 0,$$

where  $\mathbb{E}_{\frac{1}{n}}: N_{\frac{1}{n}} \rightarrow M$  is the canonical  $\psi_{\frac{1}{n}}$ -preserving normal faithful conditional expectation associated with  $J_{\frac{1}{n}}$ . For  $t \in \mathbb{Q}$ , we define the operator  $S_{\frac{1}{n}, t}: N_{\frac{1}{n}} \rightarrow N_{\frac{1}{n}}$  by

$$S_{\frac{1}{n}, t} = \begin{cases} (S_{\frac{1}{n}})^{nt} & \text{if } nt \in \mathbb{Z} \\ \text{Id}_{N_{\frac{1}{n}}} & \text{if } nt \notin \mathbb{Z}. \end{cases}$$

If  $B = \{t_1, \dots, t_k\} \subset \mathbb{Q}^+$  is a finite subset, then for  $t \in B$  and  $n \in U_B$  we have  $nt \in \mathbb{Z}$  and thus

$$(3.2) \quad T_t = (T_{\frac{1}{n}})^{nt} = \mathbb{E}_{\frac{1}{n}}(S_{\frac{1}{n}})^{nt} J_{\frac{1}{n}} = \mathbb{E}_{\frac{1}{n}} S_{\frac{1}{n}, t} J_{\frac{1}{n}},$$

i.e. the following diagram commutes.

$$\begin{array}{ccc} N_{\frac{1}{n}} & \xrightarrow{S_{\frac{1}{n}, t}} & N_{\frac{1}{n}} \\ J_{\frac{1}{n}} \uparrow & & \downarrow \mathbb{E}_{\frac{1}{n}} \\ M & \xrightarrow{T_t} & M \end{array}$$

We consider the following ultraproducts of von Neumann algebras

$$M^{\mathcal{U}} = (M, \phi)^{\mathcal{U}} \quad \text{and} \quad \tilde{N} = (N_{\frac{1}{n}}, \psi_{\frac{1}{n}})^{\mathcal{U}}.$$

We equip  $\tilde{N}$  with the normal faithful state  $\psi = (\psi_{\frac{1}{n}})^{\mathcal{U}}$ . Let  $\mathcal{I}$  be the canonical inclusion  $\mathcal{I}: M \rightarrow M^{\mathcal{U}}$   $x \mapsto (x, x, \dots)^{\mathcal{U}}$  and let  $\mathbb{E}: M^{\mathcal{U}} \rightarrow M$ ,  $(x_n)^{\mathcal{U}} \mapsto \lim_{n \rightarrow \mathcal{U}} x_n$  be the conditional expectation associated with the canonical inclusion  $\mathcal{I}: M \rightarrow M^{\mathcal{U}}$ . We introduce the operators

$$\tilde{J} = (J_{\frac{1}{n}})^{\mathcal{U}} \mathcal{I}, \quad \tilde{S}_t = (S_{\frac{1}{n}, t})^{\mathcal{U}}, \quad t \in \mathbb{Q}.$$

Observe that the map  $\tilde{J}: M \rightarrow \tilde{N}$  is a  $(\phi, \psi)$ -Markov  $*$ -monomorphism. For any  $t \in \mathbb{Q}$ , note also that the map  $\tilde{S}_t: \tilde{N} \rightarrow \tilde{N}$  is a  $*$ -automorphism of  $\tilde{N}$  (hence  $w^*$ -continuous) leaving  $\psi$  invariant. Let  $\tilde{\mathbb{E}}: \tilde{N} \rightarrow M$  be the canonical  $\psi$ -preserving faithful normal conditional expectation associated with  $\tilde{J}$ . We have  $\tilde{\mathbb{E}} = \mathbb{E} \circ (\mathbb{E}_{\frac{1}{n}})^{\mathcal{U}}$ .

Let us check that

$$\begin{array}{ccc} \tilde{S}: & \mathbb{Q} & \longrightarrow B(\tilde{N}) \\ & t & \longmapsto \tilde{S}_t \end{array}$$

is a representation and that it defines a dilation of the semigroup  $(T_t)_{t \in \mathbb{Q}^+}$ . If  $t, t' \in \mathbb{Q}$ ,  $x = (x_n)^{\mathcal{U}} \in \tilde{N}$  and  $n \in U_{\{t, t'\}}$  (i.e. for  $n$  sufficiently large) then we have  $nt, nt' \in \mathbb{Z}$  and  $n(t+t') = nt + nt' \in \mathbb{Z}$ . Then we obtain

$$S_{\frac{1}{n}, t+t'}(x_n) = (S_{\frac{1}{n}})^{n(t+t')}(x_n) = (S_{\frac{1}{n}})^{nt+nt'}(x_n) = S_{\frac{1}{n}}^{nt}(S_{\frac{1}{n}}^{nt'}(x_n)) = S_{\frac{1}{n}, t}(S_{\frac{1}{n}, t'}(x_n)).$$

We have  $(S_{n,t+t'}(x_n))^{\mathcal{U}} = (S_{n,t}(S_{n,t'}(x_n))^{\mathcal{U}})^{\mathcal{U}}$  and thus

$$\tilde{S}_{t+t'}((x_n)^{\mathcal{U}}) = \tilde{S}_t \tilde{S}_{t'}((x_n)^{\mathcal{U}}).$$

Moreover, for  $t \in \mathbb{Q}^+$  and  $x \in M$ , we have

$$\tilde{\mathbb{E}} \tilde{S}_t \tilde{J} x = \mathbb{E}(\mathbb{E}_{\frac{1}{n}})^{\mathcal{U}} (S_{\frac{1}{n},t})^{\mathcal{U}} (J_{\frac{1}{n}})^{\mathcal{U}} \mathcal{I} x = \mathbb{E}(\mathbb{E}_{\frac{1}{n}})^{\mathcal{U}} (S_{\frac{1}{n},t})^{\mathcal{U}} (J_{\frac{1}{n}})^{\mathcal{U}} \mathcal{I} x = \lim_{n \rightarrow \mathcal{U}} (\mathbb{E}_{\frac{1}{n}} S_{\frac{1}{n},t} J_{\frac{1}{n}} x)^{\mathcal{U}}.$$

By (3.2), if  $n \in U_{\{t\}}$ , we have  $\mathbb{E}_{\frac{1}{n}} S_{\frac{1}{n},t} J_{\frac{1}{n}} x = T_t x$ . We deduce that

$$\tilde{\mathbb{E}} \tilde{S}_t \tilde{J} = T_t, \quad t \in \mathbb{Q}^+.$$

We define  $\mathcal{S}$  to be the semigroup  $\bigcap_{V \in \mathcal{V}(0)} \overline{\{\tilde{S}_t : t \in V\}}^{w^*}$ . From Theorem 3.4, we deduce that the kernel of the weak\* closure  $\overline{\text{co } \mathcal{S}} = \mathcal{S}^{w^*}((\tilde{S}_t)_{t \in \mathbb{Q}})$  of the convex hull  $\text{co}(\mathcal{S})$  of  $\mathcal{S}$  is a singleton  $\{\mathbb{E}'\}$  where  $\mathbb{E}' : \tilde{N} \rightarrow \tilde{N}$  a faithful normal conditional expectation preserving  $\varphi_{\mathcal{U}}$  satisfying

$$\text{Ran } \mathbb{E}' = \left\{ x \in \tilde{N} : T(x) = x \text{ for any } T \in \mathcal{S} \right\}.$$

By Proposition 3.3, the subspace  $\tilde{N}_{w^*}$  of weak\* continuously translating elements of  $\tilde{M}$  of the representation  $\mathbb{Q} \rightarrow B(\tilde{N})$ ,  $t \mapsto \tilde{S}_t$  is equal to the fixed point subspace of  $\mathcal{S}$ :

$$\tilde{N}_{w^*} = \left\{ x \in \tilde{N} : T(x) = x \text{ for any } T \in \mathcal{S} \right\}.$$

Hence the von Neumann algebra  $\text{Ran } \mathbb{E}'$  is equal to  $\tilde{N}_{w^*}$  and is invariant under the operator  $\tilde{S}_t$  for any  $t \in \mathbb{Q}$  by Proposition 3.2. By Proposition 3.6, the range  $\text{Ran}(\tilde{J})$  of the map  $\tilde{J} : M \rightarrow \tilde{N}$  is contained in the subspace  $\tilde{N}_{w^*}$  of continuously translating elements of  $\tilde{N}$  of the representation  $\mathbb{Q} \rightarrow B(\tilde{N})$ ,  $t \mapsto \tilde{S}_t$ . Now, it is easy to obtain (1.1) by letting  $N = \tilde{N}_{w^*}$  and

$$U_t = \mathbb{E}' \tilde{S}_{t|_{\tilde{N}_{w^*}}} \quad \text{for all } t \in \mathbb{Q}.$$

where we consider  $\mathbb{E}'$  as an operator from  $N$  on  $\tilde{N}_{w^*}$ . Finally, we let  $J : M \rightarrow \tilde{N}_{w^*}$  the canonical \*-monomorphism and  $\mathbb{E} : \tilde{N}_{w^*} \rightarrow M$  the associated conditional expectation. We conclude that

$$T_t = \mathbb{E} U_t J, \quad t \geq 0.$$

■

**Remark 3.8** We refer to [AHW], [AnH], [CL] and [Oza] for QWEP von Neumann algebras. We say [Arh2, Definition 1.2] that  $T$  is QWEP-factorizable if the definition of factorizability of the introduction is satisfied with a QWEP von Neumann algebra  $N$ . Similarly, we say [Arh2, Definition 1.3] that  $(T_t)_{t \geq 0}$  is QWEP-dilatable if the definition 1.3 is satisfied with with a QWEP von Neumann algebra  $N$ . It is easy to see that if each  $T_t$  is QWEP-factorizable then the semigroup  $(T_t)_{t \geq 0}$  is QWEP-dilatable. Indeed, the proof [HaM, Theorem 4.4] of gives a QWEP von Neumann algebra (use [Oza, Proposition 4.1 (ii)b and (iii)]) and note that if each  $N_{\frac{1}{n}}$  has QWEP, then  $\tilde{N}$  has QWEP (use the proof of [AHW, Lemma 4.3] and [Oza, Proposition 4.1 (ii)]).

## 4 Dilations of semigroups on noncommutative $L^p$ -spaces

The goal is to prove Theorem 4.4 below which is a noncommutative  $L^p$  variant of Theorem 3.7. Suppose  $1 < p < \infty$ . Recall the definition of [JLM, page 239] which says that a contraction  $T: L^p(M) \rightarrow L^p(M)$  on a noncommutative  $L^p$ -space  $L^p(M)$  is dilatable if there exist a noncommutative  $L^p$ -space  $L^p(N)$ , two contractions  $J: L^p(M) \rightarrow L^p(N)$  and  $P: L^p(N) \rightarrow L^p(M)$  and an isometry  $U: L^p(N) \rightarrow L^p(N)$  such that  $T^k = PU^k J$  for any  $k \geq 0$ . Now, we introduce a variant.

**Definition 4.1** Suppose  $1 \leq p < \infty$ . We say that a completely positive contraction  $T: L^p(M) \rightarrow L^p(M)$  on a noncommutative  $L^p$ -space  $L^p(M)$  is completely positively dilatable if there exist a noncommutative  $L^p$ -space  $L^p(N)$ , two completely positive contractions  $J: L^p(M) \rightarrow L^p(N)$  and  $P: L^p(N) \rightarrow L^p(M)$  and a completely positive invertible isometry  $U: L^p(N) \rightarrow L^p(N)$  such that

$$T^k = PU^k J, \quad k \geq 0.$$

**Remark 4.2** Note that a dilatable  $\phi$ -Markov  $T: M \rightarrow M$  on a von Neumann algebra  $M$  equipped with a state  $\phi$  induces a completely positively dilatable contraction on the associated noncommutative  $L^p$ -space  $L^p(M)$ .

In this section, we use Banach ultraproducts. The same method that the beginning of the proof of Theorem 3.7 with the stability of the class of noncommutative  $L^p$ -spaces under Banach ultraproducts [Ray1] gives the following result.

**Lemma 4.3** Suppose that  $(T_t)_{t \geq 0}$  is a (not necessarily strongly continuous) semigroup of contractions on  $L^p(M)$  such that each operator  $T_t$  is completely positively dilatable. Then there exists a noncommutative  $L^p$ -space  $L^p(\tilde{N})$ , a group  $(U_t)_{t \in \mathbb{Q}}$  of completely positive invertible isometries of  $L^p(\tilde{N})$  and two completely positive contractions  $\tilde{J}: L^p(M) \rightarrow L^p(\tilde{N})$  and  $\tilde{P}: L^p(\tilde{N}) \rightarrow L^p(M)$  such that

$$T_t = \tilde{P}\tilde{U}_t\tilde{J}, \quad t \in \mathbb{Q}^+.$$

Moreover, if  $M$  has QWEP, then  $\tilde{N}$  has QWEP.

One more time, if the semigroup  $(T_t)_{t \geq 0}$  is strongly continuous, the above ultraproduct construction yields a too big space  $L^p(\tilde{N})$  such that one can expect the representation  $\tilde{U}: t \mapsto \tilde{U}_t$  of  $\mathbb{Q}$  to be continuous on  $L^p(\tilde{N})$ . However, it is still possible to restrict  $t \mapsto \tilde{U}_t$  to a smaller subspace on which the desired continuity holds. Again, the method of [Fen1] does not apply to our context.

**Theorem 4.4** Suppose  $1 < p < \infty$ . Let  $(T_t)_{t \geq 0}$  be a strongly continuous semigroup of completely positive contractions on a noncommutative  $L^p$ -space  $L^p(M)$  such that each  $T_t: L^p(M) \rightarrow L^p(M)$  is completely positively dilatable. Then there exists a noncommutative  $L^p$ -space  $L^p(N)$ , a strongly continuous group of completely positive isometries  $U_t: L^p(N) \rightarrow L^p(N)$  and two completely positive contractions  $J: L^p(M) \rightarrow L^p(N)$  and  $P: L^p(N) \rightarrow L^p(M)$  such that

$$(4.1) \quad T_t = PU_t J, \quad t \geq 0.$$

Moreover, if  $M$  has QWEP, then  $N$  has QWEP.

*Proof* : By Lemma 4.3, we obtain a representation  $\tilde{U}: \mathbb{Q} \rightarrow B(L^p(\tilde{N}))$  by completely positive isometric operators. We have

$$T_t = \tilde{P}\tilde{U}_t\tilde{J}, \quad t \in \mathbb{Q}^+.$$

By Theorem 2.1, we deduce that the kernel  $\mathcal{K}(\tilde{U})$  of  $\mathcal{S}^c(\tilde{U})$  contains a unique element  $Q_c$  and that  $Q_c: L^p(\tilde{N}) \rightarrow L^p(\tilde{N})$  is the projection from the Banach space  $L^p(\tilde{N})$  onto the subspace  $L^p(\tilde{N})_c$  of continuously translating elements. Furthermore, we have

$$\tilde{U}_t Q_c = Q_c \tilde{U}_t, \quad t \in \mathbb{Q}.$$

Thus the range  $L^p(\tilde{N})_c$  of the projection  $Q_c$  is invariant under the operator  $\tilde{U}_t$  for any  $t \in \mathbb{Q}$ . Moreover, we infer from Lemma 3.5 that the range  $\text{Ran}(\tilde{J})$  of the map  $\tilde{J}$  given by Lemma 4.3 is contained in the subspace  $L^p(\tilde{N})_c$  of continuously translating elements of  $L^p(\tilde{N})$  of the representation  $\tilde{U}$ . Furthermore, since each operator  $\tilde{U}_t$  is isometric and completely positive, hence contractive, we see that the convex semigroup  $\mathcal{S}^c(\tilde{U})$  of  $\tilde{U}$  over the identity consists of completely positive contractions only. It follows that  $Q_c$  is also contractive and completely positive and consequently that the subspace  $L^p(\tilde{N})_c$  is 1-completely positively complemented in  $L^p(\tilde{N})$ , hence a noncommutative  $L^p$ -space  $L^p(N)$  by the main result of [ArR]. Now, we define

$$(4.2) \quad U_t = Q_c \tilde{U}_{t|L^p(\tilde{N})_c} \quad \text{for all } t \in \mathbb{Q}$$

where we consider  $Q_c$  as an operator from  $L^p(\tilde{N})$  on  $L^p(\tilde{N})_c$ . Finally, we let  $J: L^p(M) \rightarrow L^p(\tilde{N})_c$  be the canonical embedding of  $L^p(M)$  into  $L^p(\tilde{N})_c$  and  $P = \tilde{P}|_{L^p(\tilde{N})_c}$ . We conclude that

$$(4.3) \quad T_t = P U_t J, \quad t \geq 0.$$

It follows from the complete positivity of the projection  $Q_c$  and from (4.2) that the induced isometry  $U_t: L^p(N) \rightarrow L^p(N)$  is also completely positive.  $\blacksquare$

**Remark 4.5** Note that there exists some completely positive contractive map  $T: S^p \rightarrow S^p$  which does not admit an isometric dilation on a noncommutative  $L^p$ -space, see [JLM]. See also [Arh1] and [ALM] for more information on dilations on noncommutative  $L^p$ -spaces.

## 5 Semigroups of selfadjoint Fourier multipliers

As we said in the introduction, Haagerup and Musat [HaM, Theorem 4.4] have characterised dilatable Markov maps. Indeed, they proved that if  $T: M \rightarrow M$  is a  $\phi$ -Markov map on a von Neumann algebra  $M$  equipped with a state  $\phi$  then  $T$  is dilatable if and only if  $T$  is factorizable in the sense of [AnD]. This result allows us to give concrete examples of dilatable semigroups.

Suppose that  $G$  is a discrete group. We denote by  $e_G$  the neutral element of  $G$ . We denote by  $\lambda_g: \ell_G^2 \rightarrow \ell_G^2$  the unitary operator of left translation by  $g$  and  $\text{VN}(G)$  the von Neumann algebra of  $G$  spanned by the  $\lambda_g$ 's where  $g \in G$ . It is a finite algebra with its canonical faithful normal finite trace given by

$$\tau_G(x) = \langle \varepsilon_{e_G}, x(\varepsilon_{e_G}) \rangle_{\ell_G^2}$$

where  $(\varepsilon_g)_{g \in G}$  is the canonical basis of  $\ell_G^2$  and  $x \in \text{VN}(G)$ . A Fourier multiplier is a normal linear map  $T: \text{VN}(G) \rightarrow \text{VN}(G)$  such that there exists a complex function  $t: G \rightarrow \mathbb{C}$  such that  $T(\lambda_g) = t_g \lambda_g$  for any  $g \in G$ . In this case, we denote  $T$  by  $M_t: \text{VN}(G) \rightarrow \text{VN}(G)$ . It is well-known that a Fourier multiplier  $M_t: \text{VN}(G) \rightarrow \text{VN}(G)$  is completely positive if and only if the function  $t$  is positive definite. It is easy to see that a  $\tau_G$ -Markov Fourier multiplier  $M_t: \text{VN}(G) \rightarrow \text{VN}(G)$  is selfadjoint if and only if  $t: G \rightarrow \mathbb{C}$  is a real function.

Using the factorisability of selfadjoint  $\tau_G$ -Markov Fourier multipliers of [Ric], we deduce the following result:

**Corollary 5.1** *Let  $G$  be a discrete group. Let  $(T_t)_{t \geq 0}$  be a  $w^*$ -semigroup of selfadjoint  $\tau_G$ -Markov Fourier multipliers on the von Neumann algebra  $\text{VN}(G)$ . Then the semigroup  $(T_t)_{t \geq 0}$  is dilatable.*

## 6 Applications to $H^\infty$ functional calculus

We start with a little background on sectoriality and  $H^\infty$  functional calculus. We refer to [Haa], [KW], [JMX] and [Arh2] for details and complements. Let  $X$  be a Banach space. A closed densely defined linear operator  $A: D(A) \subset X \rightarrow X$  is called sectorial of type  $\omega$  if its spectrum  $\sigma(A)$  is included in the closed sector  $\overline{\Sigma_\omega}$ , and for any angle  $\omega < \theta < \pi$ , there is a positive constant  $K_\theta$  such that

$$\|(\lambda - A)^{-1}\|_{X \rightarrow X} \leq \frac{K_\theta}{|\lambda|}, \quad \lambda \in \mathbb{C} - \overline{\Sigma_\theta}.$$

If  $-A$  is the negative generator of a bounded strongly continuous semigroup on a  $X$  then  $A$  is sectorial of type  $\frac{\pi}{2}$ . We also recall that sectorial operators of type  $< \frac{\pi}{2}$  coincide with negative generators of bounded analytic semigroups.

For any  $0 < \theta < \pi$ , let  $H^\infty(\Sigma_\theta)$  be the algebra of all bounded analytic functions  $f: \Sigma_\theta \rightarrow \mathbb{C}$ , equipped with the supremum norm  $\|f\|_{H^\infty(\Sigma_\theta)} = \sup\{|f(z)| : z \in \Sigma_\theta\}$ . Let  $H_0^\infty(\Sigma_\theta) \subset H^\infty(\Sigma_\theta)$  be the subalgebra of bounded analytic functions  $f: \Sigma_\theta \rightarrow \mathbb{C}$  for which there exist  $s, c > 0$  such that  $|f(z)| \leq c|z|^s(1 + |z|)^{-2s}$  for any  $z \in \Sigma_\theta$ .

Given a sectorial operator  $A$  of type  $0 < \omega < \pi$ , a bigger angle  $\omega < \theta < \pi$ , and a function  $f \in H_0^\infty(\Sigma_\theta)$ , one may define a bounded operator  $f(A)$  by means of a Cauchy integral (see e.g. [Haa, Section 2.3] or [KW, Section 9]). The resulting mapping  $H_0^\infty(\Sigma_\theta) \rightarrow B(X)$  taking  $f$  to  $f(A)$  is an algebra homomorphism. By definition,  $A$  has a bounded  $H^\infty(\Sigma_\theta)$  functional calculus provided that this homomorphism is bounded, that is, there exists a positive constant  $C$  such that  $\|f(A)\|_{X \rightarrow X} \leq C\|f\|_{H^\infty(\Sigma_\theta)}$  for any  $f \in H_0^\infty(\Sigma_\theta)$ . In the case when  $A$  has a dense range, the latter boundedness condition allows a natural extension of  $f \mapsto f(A)$  to the full algebra  $H^\infty(\Sigma_\theta)$ .

Note a noncommutative  $L^p$ -space is a UMD Banach space [PiX, Corollary 7.7]. Using the connection between the existence of dilations in UMD spaces and  $H^\infty$  functional calculus [KW, Corollary 10.9], Theorem 4.4 and the angle reduction principle of [JMX, Proposition 5.8] we obtain:

**Theorem 6.1** *Let  $M$  be a von Neumann algebra equipped with a normal faithful state  $\phi$ . Let  $(T_t)_{t \geq 0}$  be a  $w^*$ -semigroup of factorizable  $\phi$ -Markov maps on  $M$ . Suppose  $1 < p < \infty$ . We let  $-A_p$  be the generator of the induced strongly continuous semigroup  $(T_t)_{t \geq 0}$  on the Banach space  $L^p(M)$ . Then for any  $\theta > \pi|\frac{1}{p} - \frac{1}{2}|$ , the operator  $A_p$  has a bounded  $H^\infty(\Sigma_\theta)$  functional calculus.*

This result give a partial answer to the question of [JMX, page 57]. This theorem is applicable to any  $w^*$ -semigroup  $(T_t)_{t \geq 0}$  of selfadjoint  $\tau_G$ -Markov Fourier multipliers on the von Neumann algebra  $\text{VN}(G)$  of a discrete group  $G$ .

Using Remark 3.8, we can also simplify the vector-valued variant of [Arh2, Theorem 1.4]. Here, we use the notations from [Arh2].

**Theorem 6.2** *Let  $M$  be a von Neumann algebra with QWEP equipped with a normal faithful state. Let  $(T_t)_{t \geq 0}$  be a  $w^*$ -continuous semigroup of QWEP-factorizable  $\phi$ -Markov on  $M$ . Suppose  $1 < p, q < \infty$  and  $0 < \alpha < 1$ . Let  $E$  be an operator space such that  $E = (\text{OH}(I), F)_\alpha$*

for some index set  $I$  and for some  $\text{OUMD}'_q$  operator space  $F$  with  $\frac{1}{p} = \frac{1-\alpha}{2} + \frac{\alpha}{q}$ . We let  $-A_p$  be the generator of the strongly continuous semigroup  $(T_t \otimes \text{Id}_E)_{t \geq 0}$  on the vector valued noncommutative  $L^p$ -space  $L^p(M, E)$ . Then for some  $0 < \theta < \frac{\pi}{2}$ , the operator  $A_p$  has a completely bounded  $H^\infty(\Sigma_\theta)$  functional calculus.

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